Problem 3.40

The most general wave function of a particle in the simple harmonic oscillator potential is

$$\Psi(x,t) = \sum_n c_n \psi_n(x) e^{-iE_n t/\hbar}.$$

Show that the expectation value of position is

$$\langle x \rangle = C\cos(\omega t - \phi),$$

where the real constants C and ϕ are given by

$$Ce^{-i\phi} = \left(\sqrt{\frac{2\hbar}{m\omega}}\right) \sum_{n=0}^{\infty} \sqrt{n+1}\, c_{n+1}^* c_n.$$

Thus the expectation value of position for a particle in the harmonic oscillator oscillates at the classical frequency ω (as you would expect from Ehrenfest's theorem; see problem 3.19(b)). *Hint:* Use Equation 3.114. As an example, find C and ϕ for the wave function in Problem 2.40.

[Capitalize the "p" to be consistent.]

Solution

Method 1

Use the method of Example 2.5 on page 47 and express the position operator in terms of the promotion and demotion operators, $\hat{a}_+$ and $\hat{a}_-$, respectively.

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\,(\hat{a}_+ + \hat{a}_-)$$

Representing the nth eigenstate $\psi_n(x)$ of the harmonic oscillator as a ket $|n\rangle$, the promotion and demotion operators satisfy

$$\hat{a}_+|n\rangle = \sqrt{n+1}\,|n+1\rangle$$
$$\hat{a}_-|n\rangle = \sqrt{n}\,|n-1\rangle.$$

So then

$$\begin{aligned}
\langle x \rangle &= \langle \Psi \,|\, \hat{x} \,|\, \Psi \rangle \\
&= \left(\sum_{q=0}^{\infty} c_q^* e^{iE_q t/\hbar} \langle q|\right) \hat{x} \left(\sum_{n=0}^{\infty} c_n e^{-iE_n t/\hbar}|n\rangle\right) \\
&= \sum_{q=0}^{\infty}\sum_{n=0}^{\infty} c_q^* c_n e^{i(E_q - E_n)t/\hbar} \langle q \,|\, \hat{x} \,|\, n \rangle \\
&= \sum_{q=0}^{\infty}\sum_{n=0}^{\infty} c_q^* c_n e^{i(E_q - E_n)t/\hbar} \langle q| \cdot (\hat{x}|n\rangle)\,.
\end{aligned}$$

Use the fact that the eigenstates of the harmonic oscillator are orthonormal: $\langle q \,|\, n\rangle = \delta_{qn}$.

$$\begin{aligned}
\langle x\rangle &= \sum_{q=0}^{\infty}\sum_{n=0}^{\infty} c_q^* c_n e^{i(E_q-E_n)t/\hbar}\langle q|\cdot\left[\sqrt{\frac{\hbar}{2m\omega}}\,(\hat{a}_+ + \hat{a}_-)\,|n\rangle\right] \\
&= \sqrt{\frac{\hbar}{2m\omega}}\sum_{q=0}^{\infty}\sum_{n=0}^{\infty} c_q^* c_n e^{i(E_q-E_n)t/\hbar}\langle q|\cdot(\hat{a}_+|n\rangle + \hat{a}_-|n\rangle) \\
&= \sqrt{\frac{\hbar}{2m\omega}}\sum_{q=0}^{\infty}\sum_{n=0}^{\infty} c_q^* c_n e^{i(E_q-E_n)t/\hbar}\langle q|\cdot\left(\sqrt{n+1}\,|n+1\rangle + \sqrt{n}\,|n-1\rangle\right) \\
&= \sqrt{\frac{\hbar}{2m\omega}}\sum_{q=0}^{\infty}\sum_{n=0}^{\infty} c_q^* c_n e^{i(E_q-E_n)t/\hbar}\left(\sqrt{n+1}\,\langle q\,|\,n+1\rangle + \sqrt{n}\,\langle q\,|\,n-1\rangle\right) \\
&= \sqrt{\frac{\hbar}{2m\omega}}\sum_{q=0}^{\infty}\sum_{n=0}^{\infty} c_q^* c_n e^{i(E_q-E_n)t/\hbar}\left(\sqrt{n+1}\,\delta_{q,n+1} + \sqrt{n}\,\delta_{q,n-1}\right) \\
&= \sqrt{\frac{\hbar}{2m\omega}}\sum_{q=0}^{\infty}\sum_{n=0}^{\infty} c_q^* c_n e^{i(E_q-E_n)t/\hbar}\left(\sqrt{n+1}\,\delta_{q,n+1} + \sqrt{n}\,\delta_{q+1,n}\right) \\
&= \sqrt{\frac{\hbar}{2m\omega}}\left[\sum_{q=0}^{\infty}\sum_{n=0}^{\infty} c_q^* c_n e^{i(E_q-E_n)t/\hbar}\sqrt{n+1}\,\delta_{q,n+1} + \sum_{q=0}^{\infty}\sum_{n=0}^{\infty} c_q^* c_n e^{i(E_q-E_n)t/\hbar}\sqrt{n}\,\delta_{q+1,n}\right] \\
&= \sqrt{\frac{\hbar}{2m\omega}}\left[\sum_{n=0}^{\infty} c_{n+1}^* c_n e^{i(E_{n+1}-E_n)t/\hbar}\sqrt{n+1} + \sum_{q=0}^{\infty} c_q^* c_{q+1} e^{i(E_q-E_{q+1})t/\hbar}\sqrt{q+1}\right]
\end{aligned}$$

q is just a dummy index and can be replaced with n.

$$\begin{aligned}
\langle x\rangle &= \sqrt{\frac{\hbar}{2m\omega}}\left[\sum_{n=0}^{\infty} c_{n+1}^* c_n e^{i(E_{n+1}-E_n)t/\hbar}\sqrt{n+1} + \sum_{n=0}^{\infty} c_n^* c_{n+1} e^{i(E_n-E_{n+1})t/\hbar}\sqrt{n+1}\right] \\
&= \sqrt{\frac{\hbar}{2m\omega}}\sum_{n=0}^{\infty}\sqrt{n+1}\left[c_{n+1}^* c_n e^{i(E_{n+1}-E_n)t/\hbar} + c_n^* c_{n+1} e^{i(E_n-E_{n+1})t/\hbar}\right] \\
&= 2\sqrt{\frac{\hbar}{2m\omega}}\sum_{n=0}^{\infty}\sqrt{n+1}\left\{\frac{c_{n+1}^* c_n e^{i(E_{n+1}-E_n)t/\hbar} + \left[c_{n+1}^* c_n e^{i(E_{n+1}-E_n)t/\hbar}\right]^*}{2}\right\} \\
&= \sqrt{\frac{2\hbar}{m\omega}}\sum_{n=0}^{\infty}\sqrt{n+1}\,\mathrm{Re}\ c_{n+1}^* c_n e^{i(E_{n+1}-E_n)t/\hbar}
\end{aligned}$$

Note that for a complex number z, Re $z = (z+z^*)/2$. The energy of the nth eigenstate of the harmonic oscillator is

$$E_n = \left(n+\frac{1}{2}\right)\hbar\omega,$$

so $E_{n+1} - E_n = \hbar\omega$.

As a result,

$$\begin{aligned}\langle x\rangle &= \sqrt{\frac{2\hbar}{m\omega}}\sum_{n=0}^{\infty}\sqrt{n+1}\,\text{Re}\ c_{n+1}^{*}c_n e^{i\omega t}\\ &= \text{Re}\ e^{i\omega t}\left(\sqrt{\frac{2\hbar}{m\omega}}\sum_{n=0}^{\infty}\sqrt{n+1}\,c_{n+1}^{*}c_n\right).\end{aligned}$$

Define the complex number in parentheses to be $Ce^{-i\phi}$, where C and ϕ are real numbers.

$$\begin{aligned}\langle x\rangle &= \text{Re}\ e^{i\omega t}(Ce^{-i\phi})\\ &= C\,\text{Re}\ e^{i(\omega t-\phi)}\end{aligned}$$

Therefore,

$$\langle x\rangle = C\cos(\omega t-\phi).$$

Method 2

This is another way to prove the result, using the boxed formula for $\langle n \,|\, \hat{x} \,|\, n' \rangle$ in Problem 3.39.

$$\langle n \,|\, \hat{x} \,|\, n' \rangle = \sqrt{\frac{\hbar}{2m\omega}} \left(\sqrt{n}\, \delta_{n,n'+1} + \sqrt{n'}\, \delta_{n',n+1} \right)$$

Consequently,

$$\begin{aligned}
\langle x \rangle &= \langle \Psi \,|\, \hat{x} \,|\, \Psi \rangle \\
&= \langle \Psi \,|\, \hat{I}\hat{x}\hat{I} \,|\, \Psi \rangle \\
&= \langle \Psi | \left(\sum_{j=0}^{\infty} |j\rangle\langle j| \right) \hat{x} \left(\sum_{k=0}^{\infty} |k\rangle\langle k| \right) |\Psi\rangle \\
&= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \langle \Psi \,|\, j \rangle \langle j \,|\, \hat{x} \,|\, k \rangle \langle k \,|\, \Psi \rangle \\
&= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \langle \Psi \,|\, j \rangle \left[\sqrt{\frac{\hbar}{2m\omega}} \left(\sqrt{j}\, \delta_{j,k+1} + \sqrt{k}\, \delta_{k,j+1} \right) \right] \langle k \,|\, \Psi \rangle \\
&= \sqrt{\frac{\hbar}{2m\omega}} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \langle \Psi \,|\, j \rangle \left(\sqrt{j}\, \delta_{j,k+1} + \sqrt{k}\, \delta_{k,j+1} \right) \langle k \,|\, \Psi \rangle \\
&= \sqrt{\frac{\hbar}{2m\omega}} \left(\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sqrt{j}\, \langle \Psi \,|\, j \rangle \langle k \,|\, \Psi \rangle \delta_{j,k+1} + \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sqrt{k}\, \langle \Psi \,|\, j \rangle \langle k \,|\, \Psi \rangle \delta_{k,j+1} \right) \\
&= \sqrt{\frac{\hbar}{2m\omega}} \left(\sum_{k=0}^{\infty} \sqrt{k+1}\, \langle \Psi \,|\, k+1 \rangle \langle k \,|\, \Psi \rangle + \sum_{j=0}^{\infty} \sqrt{j+1}\, \langle \Psi \,|\, j \rangle \langle j+1 \,|\, \Psi \rangle \right).
\end{aligned}$$

j is just a dummy index and can be replaced with k.

$$\begin{aligned}
&= \sqrt{\frac{\hbar}{2m\omega}} \left(\sum_{k=0}^{\infty} \sqrt{k+1}\, \langle \Psi \,|\, k+1 \rangle \langle k \,|\, \Psi \rangle + \sum_{k=0}^{\infty} \sqrt{k+1}\, \langle \Psi \,|\, k \rangle \langle k+1 \,|\, \Psi \rangle \right) \\
&= \sqrt{\frac{\hbar}{2m\omega}} \sum_{k=0}^{\infty} \sqrt{k+1} \left(\langle \Psi \,|\, k+1 \rangle \langle k \,|\, \Psi \rangle + \langle \Psi \,|\, k \rangle \langle k+1 \,|\, \Psi \rangle \right) \\
&= 2\sqrt{\frac{\hbar}{2m\omega}} \sum_{k=0}^{\infty} \sqrt{k+1} \left[\frac{\langle \Psi \,|\, k+1 \rangle \langle k \,|\, \Psi \rangle + (\langle \Psi \,|\, k+1 \rangle \langle k \,|\, \Psi \rangle)^*}{2} \right] \\
&= \sqrt{\frac{2\hbar}{m\omega}} \sum_{k=0}^{\infty} \sqrt{k+1}\, \mathrm{Re}\, \langle \Psi \,|\, k+1 \rangle \langle k \,|\, \Psi \rangle
\end{aligned}$$

Substitute the formula for $\Psi(x,t)$.

$$\begin{aligned}
\langle x\rangle &= \sqrt{\frac{2\hbar}{m\omega}}\sum_{k=0}^{\infty}\sqrt{k+1}\,\mathrm{Re}\,\left(\sum_{q=0}^{\infty}c_q^* e^{iE_q t/\hbar}\langle q|\right)|k+1\rangle\langle k|\left(\sum_{n=0}^{\infty}c_n e^{-iE_n t/\hbar}|n\rangle\right)\\
&= \sqrt{\frac{2\hbar}{m\omega}}\sum_{k=0}^{\infty}\sqrt{k+1}\,\mathrm{Re}\,\sum_{q=0}^{\infty}\sum_{n=0}^{\infty}c_q^* c_n e^{i(E_q-E_n)t/\hbar}\langle q\,|\,k+1\rangle\langle k\,|\,n\rangle\\
&= \sqrt{\frac{2\hbar}{m\omega}}\sum_{k=0}^{\infty}\sqrt{k+1}\,\mathrm{Re}\,\sum_{q=0}^{\infty}\sum_{n=0}^{\infty}c_q^* c_n e^{i(E_q-E_n)t/\hbar}\delta_{q,k+1}\delta_{kn}\\
&= \sqrt{\frac{2\hbar}{m\omega}}\sum_{k=0}^{\infty}\sqrt{k+1}\,\mathrm{Re}\,\sum_{n=0}^{\infty}c_{k+1}^* c_n e^{i(E_{k+1}-E_n)t/\hbar}\delta_{kn}\\
&= \sqrt{\frac{2\hbar}{m\omega}}\sum_{n=0}^{\infty}\sqrt{n+1}\,\mathrm{Re}\;c_{n+1}^* c_n e^{i(E_{n+1}-E_n)t/\hbar}
\end{aligned}$$

The energy of the nth eigenstate of the harmonic oscillator is

$$E_n = \left(n+\frac{1}{2}\right)\hbar\omega,$$

so $E_{n+1}-E_n = \hbar\omega$.

$$\begin{aligned}
\langle x\rangle &= \sqrt{\frac{2\hbar}{m\omega}}\sum_{n=0}^{\infty}\sqrt{n+1}\,\mathrm{Re}\;c_{n+1}^* c_n e^{i\omega t}\\
&= \mathrm{Re}\;e^{i\omega t}\left(\sqrt{\frac{2\hbar}{m\omega}}\sum_{n=0}^{\infty}\sqrt{n+1}\,c_{n+1}^* c_n\right)
\end{aligned}$$

Define the complex number in parentheses to be $Ce^{-i\phi}$, where C and ϕ are real numbers.

$$\begin{aligned}
\langle x\rangle &= \mathrm{Re}\;e^{i\omega t}(Ce^{-i\phi})\\
&= C\,\mathrm{Re}\;e^{i(\omega t-\phi)}
\end{aligned}$$

Therefore,

$$\langle x\rangle = C\cos(\omega t-\phi).$$

Problem 2.40

In Problem 2.40 the wave function is

$$\begin{aligned}\Psi(x,t) &= c_0\psi_0(x)e^{-iE_0t/\hbar} + c_1\psi_1(x)e^{-iE_1t/\hbar} + c_2\psi_2(x)e^{-iE_2t/\hbar} \\ &= \frac{3}{5}\psi_0(x)e^{-i\omega t/2} - \frac{2\sqrt{2}}{5}\psi_1(x)e^{-3i\omega t/2} + \frac{2\sqrt{2}}{5}\psi_2(x)e^{-5i\omega t/2}.\end{aligned}$$

Therefore,

$$\begin{aligned}Ce^{-i\phi} &= \sqrt{\frac{2\hbar}{m\omega}}\sum_{n=0}^{\infty}\sqrt{n+1}\,c_{n+1}^*c_n \\ &= \sqrt{\frac{2\hbar}{m\omega}}\left(\sqrt{0+1}\,c_{0+1}^*c_0 + \sqrt{1+1}\,c_{1+1}^*c_1 + \underbrace{\sqrt{2+1}\,c_{2+1}^*c_2}_{=0} + \underbrace{\sqrt{3+1}\,c_{3+1}^*c_3}_{=0} + \cdots\right) \\ &= \sqrt{\frac{2\hbar}{m\omega}}\left(c_1^*c_0 + \sqrt{2}\,c_2^*c_1\right) \\ &= \sqrt{\frac{2\hbar}{m\omega}}\left[\left(-\frac{2\sqrt{2}}{5}\right)\left(\frac{3}{5}\right) + \sqrt{2}\left(\frac{2\sqrt{2}}{5}\right)\left(-\frac{2\sqrt{2}}{5}\right)\right] \\ &= \sqrt{\frac{2\hbar}{m\omega}}\left(-\frac{14\sqrt{2}}{25}\right) \\ &= -\frac{28}{25}\sqrt{\frac{\hbar}{m\omega}},\end{aligned}$$

which means

$$C = \frac{28}{25}\sqrt{\frac{\hbar}{m\omega}} \quad \text{and} \quad \phi = \pi + 2\pi n,$$

where n is any integer.